Phase transitions for the topological de Sitter spaces and Schwarzschild-de Sitter black hole

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Abstract

We study whether the Hawking-Page phase transition may occur in topological de Sitter spaces (TdS) and Schwarzschild-de Sitter black hole (SdS). We show that at the critical temperature $T = T_1$, TdS with hyperbolic cosmological horizon can make the Hawking-Page transition from the zero mass de Sitter space to TdS. It is also shown that there is no Hawking-Page transition for TdS with Ricci-flat and spherical horizons, when the zero mass de Sitter space is taken as the thermal background. Also we find that the SdS undergoes a different phase transition at $T = 0$ which the Nariai black hole is formed. Finally we connect our results to the dS/CFT correspondence.

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1 Introduction

A number of authors have shown that for a large class of black holes, the Bekenstein-Hawking entropy receives logarithmic corrections due to thermodynamic fluctuations [1]. A corrected formula takes the form

$$S_c = S - \frac{1}{2} \ln C + \cdots, \tag{1}$$

where C is the specific heat of the given system, and S denotes the uncorrected Bekenstein-Hawking entropy. Here C should be positive for Eq.(1) to be well-defined. We note that for $C > 0$ ($C < 0$), the system is thermodynamically stable (unstable). A black hole with negative specific heat is in an unstable equilibrium with the heat reservoir of the temperature T [2]. Its fate under small fluctuations will be either to decay to hot flat space or to grow without limit by absorbing thermal radiation in the heat reservoir [3]. There exists a way to achieve a stable black hole in an equilibrium with the heat reservoir. A black hole could be rendered thermodynamically stable by placing it in AdS space. An important point is to understand how a black hole with positive specific heat could emerge from thermal radiation through a phase transition. To this end, one introduces the Hawking-Page phase transition between thermal AdS space and Schwarzschild-AdS black hole [4, 5, 6].

Further, there is a close similarity between the event horizon of a black hole and the cosmological horizon of de Sitter space [7]. Hence, it is interesting to study the thermal properties of various de Sitter spaces. In this work, we check whether a Hawking-Page phase transition occurs in topological de Sitter spaces. Also we study the phase transition of the Schwarzschild-de Sitter black hole, where a black hole is inside the cosmological horizon. Moreover, we investigate the implications of Hawking-Page transition on dS/CFT correspondence.

Our study is based on the observations of heat capacity, free energy, and generalized (off-shell) free energy near the phase transition.

The organization of this work is as follows. Section 2 is devoted to reviewing topological AdS black holes. We discuss the thermodynamic properties of topological de Sitter spaces which are similar to those of topological AdS black holes in section 3. The thermodynamic properties of Schwarzschild-de Sitter black hole are investigated in section 4. In section 5 we study whether or not the Hawking-Page phase transition occurs in TdS and SdS. We connect our results to the dS/CFT correspondence in section 6 by introducing boundary CFT and Cardy-Verlinde formula. Finally, we discuss our results in section 7.
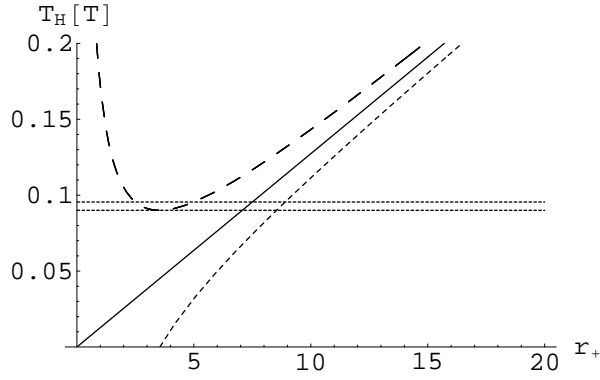
Figure 1: Temperature for TAdS with $r_+ = r_E$. The long-dashed curve denotes $T_H^{SAdS}(\ell = 5, r_+)$, showing a minimum temperature $T = T_0$ at $r_+ = r_0 = 3.5$ with units $G_5 = 1$ and $\ell = 5$. The solid line is a monotonically increasing function $T_H^{FAdS}(\ell = 5, r_+)$. The short-dashed curve represents $T_H^{HAdS}(\ell = 5, r_+ \geq r_0)$, indicating a forbidden region of $0 \leq r_+ < r_0$. Two horizontal lines denote the temperatures $T = T_0$ and T_1 for the heat reservoir.

2   Topological AdS black holes

We begin with a review of topological black holes in AdS space [8], as these are thermodynamically related to TdS [9]. A black hole in asymptotically flat spacetime has a spherical horizon. If one introduces a negative cosmological constant, a black hole can have a non-spherical horizon. This is called the topological AdS black holes (TAdS) whose metric in five dimensions is given by

$$ds^2_{TAdS} = -h(r)dt^2 + \frac{1}{h(r)}dr^2 + r^2 d\Sigma_k^2, \tag{2}$$

where $d\Sigma_k^2 = d\chi^2 + f_k(\chi)^2(d\theta^2 + \sin^2\theta d\phi^2)$ describes the horizon geometry with a constant curvature. Further $h(r)$ and $f_k(\chi)$ are given by

$$h(r) = k - \frac{m}{r^2} + \frac{r^2}{\ell^2}, \quad f_0(\chi) = \chi, \ f_1(\chi) = \sin\chi, \ f_{-1}(\chi) = \sinh\chi. \tag{3}$$

Here we define $k=1$, 0, and -1 cases as the Schwarzschild-AdS black hole (SAdS) [10, 11], flat-AdS black hole (FAdS), and hyperbolic-AdS black hole (HAdS) [12], respectively. In the case of $k = 1$ and $m = 0$, we have a thermal AdS space with curvature radius ℓ.

The location of the event horizon is given by

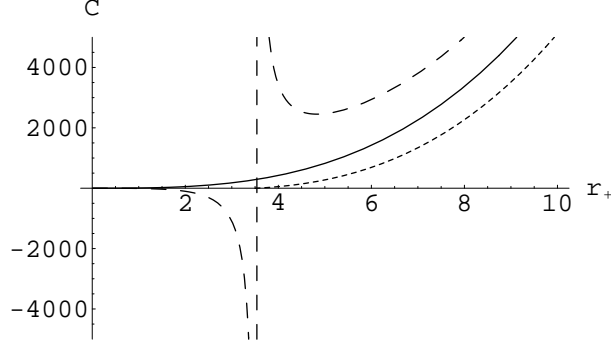$$r_E^2 = \frac{\ell^2}{2}\Big(-k + \sqrt{k^2 + 4m/\ell^2}\Big). \tag{4}$$

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Figure 2: Specific heats for TAdS. Here $r_+ = r_E$ plays the role of effective temperature. The long-dashed curve denotes $C^{SAdS}(\ell = 5, r_+)$, and diverges at $r_+ = r_0$. The solid curve represents a monotonically increasing specific heat $C^{FAdS}(\ell = 5, r_+)$. The short-dashed curve indicates $C^{HAdS}(\ell = 5, r_+ \geq r_0)$, which shows a forbidden region of $0 \leq r_+ < r_0$.
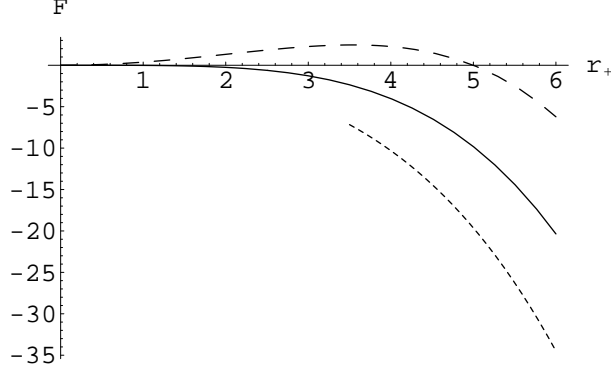

Figure 3: Three graphs of free energy for TAdS. The long-dashed curve represents $F^{SAdS}(\ell = 5, r_+)$, which shows a continuous transition from positive value to negative one at $r_+ = r_1$. The solid curve represents $F^{FAdS}(\ell = 5, r_+)$, while the short-dashed curve denotes $F^{HAdS}(\ell = 5, r_+ \geq r_0)$ with a forbidden region.

The relevant thermodynamic quantities: reduced mass (m), free energy (F), Bekenstein-Hawking entropy (S), Hawking temperature (T_H), energy (ADM mass: $E = F + T_H S = M$), and specific heat (C) are given by[1]

$$m = r_E^2\Big(\frac{r_E^2}{\ell^2} + k\Big), \quad F = -\frac{V_3 r_E^2}{16\pi G_5}\Big(\frac{r_E^2}{\ell^2} - k\Big), \quad S = \frac{V_3 r_E^3}{4G_5}, \tag{5}$$

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[1]In defining the energy E and free energy F for the HAdS of $k = -1$, we do not include $M_{\mathrm{crit}} \equiv M|_{r_+ = r_0} = -3V_3\ell^2/64\pi G_5$ of Ref.[8]. Even if we include this term, the phase transition is unaffected. This is because it is a constant [13].

$$T_H = \frac{k}{2\pi r_E} + \frac{r_E}{\pi \ell^2}, \quad E = \frac{3V_3 m}{16\pi G_5}, \quad C = 3\frac{2r_E^2 + k\ell^2}{2r_E^2 - k\ell^2}S,$$

where V_3 is the volume of a unit three-dimensional hypersurface Σ_k and G_5 is the five-dimensional Newton constant. The different behaviors of temperature are shown in Fig. 1 [14]. From the equilibrium condition ($T = T_H$) of a black hole with the heat reservoir, we find the two solutions: a small, unstable black hole of size

$$r_u = \frac{\pi \ell^2 T}{2}\left[1 - \sqrt{1 - \frac{8k}{(2\pi \ell T)^2}}\right] \tag{6}$$

and a large, stable black hole of size

$$r_s = \frac{\pi \ell^2 T}{2}\left[1 + \sqrt{1 - \frac{8k}{(2\pi \ell T)^2}}\right]. \tag{7}$$

Here we find that $r_u = 0$, $r_s = \pi \ell^2 T$ for $k = 0$; $r_u \simeq 1/2\pi T$, $r_s \simeq \pi \ell^2 T$ for $k = 1$ and $T \gg 1/\ell$; $r_s \simeq \pi \ell^2 T$ for $k = -1$ and $T \gg 1/\ell$. For $k = 1$, an extremum is at $r_u = r_s = r_0 = \ell/\sqrt{2}$ and the corresponding temperature is given by $T_0 = \sqrt{2}/\pi \ell$. Further, we obtain a critical point $r_1 = \ell$ from the condition of $F = 0$. The corresponding temperature is given by $T_1 = 3/2\pi \ell$. At this temperature, we have $r_u = \ell/2$ and $r_s = \ell$. As is shown in Fig. 2, there are three different specific heats. C^{SAdS} has a simple pole at $r_+ = r_0$, while C^{FAdS} is a continuous increasing function. In contrast, C^{HAdS} has a forbidden region which leads to the interruption of thermodynamic analysis. Finally, we observe from Fig. 3 that F^{FAdS} and F^{HAdS} are monotonically decreasing functions, but F^{HAdS} has a forbidden region. F^{SAdS} has a maximum value at $r_+ = r_0$ and is zero at $r_+ = r_1$, which shows a feature of the Hawking-Page transition.

3 Topological de Sitter spaces

The topological de Sitter space (TdS) solution was originally introduced to check the mass bound conjecture in de Sitter space: any asymptotically de Sitter space with the mass greater than an exact de Sitter space has a cosmological singularity [15]. It is very interesting to study the thermodynamic properties of this kind of cosmological horizon. We consider the topological de Sitter solution in five-dimensional spacetime

$$ds_{TdS}^2 = -h(r)dt^2 + \frac{1}{h(r)}dr^2 + r^2 d\Sigma_k^2, \tag{8}$$

where $k = 0, \pm 1$ and $h(r)$ is given by

$$h(r) = k + \frac{m}{r^2} - \frac{r^2}{\ell^2}. \tag{9}$$

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Requiring $m > 0$ leads to the fact that the event horizon of black hole disappears and instead, a naked singularity appears at $r = 0$ inside the cosmological horizon. Here we define $k = 1$, 0, and -1 cases as the Schwarzschild-topological de Sitter space (STdS), flat-topological de Sitter space (FTdS), and hyperbolic–topological de Sitter space (HTdS), respectively. In the case of $k = 1$ and $m = 0$, we have an exact de Sitter space with curvature radius ℓ. The cosmological horizon is at

$$r_C^2 = \frac{\ell^2}{2}\Big(k + \sqrt{k^2 + 4m/\ell^2}\Big). \tag{10}$$

We note that there is no restriction on r_C, in contrast with the cosmological horizon of SdS. The thermodynamic quantities for the cosmological horizon are [16]

$$m = r_C^2\Big(\frac{r_C^2}{\ell^2} - k\Big), \quad F = -\frac{V_3 r_C^2}{16\pi G_5}\Big(\frac{r_C^2}{\ell^2} + k\Big), \quad S = \frac{V_3 r_C^3}{4G_5}, \tag{11}$$

$$T_H = -\frac{k}{2\pi r_C} + \frac{r_C}{\pi \ell^2}, \quad E = \frac{3V_3 m}{16\pi G_5} = M, \quad C = 3\frac{2r_C^2 - k\ell^2}{2r_C^2 + k\ell^2}S,$$

where V_3 is the volume of a unit three-dimensional hypersurface Σ_k. Considering the equilibrium condition $T = T_H$, we find that two solutions: a small, unstable cosmological horizon of size

$$r_u = \frac{\pi \ell^2 T}{2}\left[1 - \sqrt{1 + \frac{8k}{(2\pi \ell T)^2}}\right] \tag{12}$$

and a large, stable cosmological horizon of size

$$r_s = \frac{\pi \ell^2 T}{2}\left[1 + \sqrt{1 + \frac{8k}{(2\pi \ell T)^2}}\right]. \tag{13}$$

From the above, we find $r_u = 0$, $r_s = \pi \ell^2 T$ for $k = 0$; $r_u \simeq 1/2\pi T$, $r_s \simeq \pi \ell^2 T$ for $k = -1$ and $T \gg 1/\ell$; $r_s \simeq \pi \ell^2 T$ for $k = 1$ and $T \gg 1/\ell$. For $k = -1$, there is an extremum at $r_u = r_s = r_0 = \ell/\sqrt{2}$ and $T_0 = \sqrt{2}/\pi \ell$. Further, we obtain a critical point $r_1 = \ell$ from the condition of $F = 0$. The corresponding temperature is also given by $T_1 = 3/2\pi \ell$.

All results of TdS solution may be recovered from TAdS by substituting k and r_E into $-k$ and r_C: SAdS $\to$ HTdS, HAdS $\to$ STdS, and FAdS $\to$ FTdS. Then we may obtain all thermodynamic behaviors of TdS from Figs. 1, 2 and 3.

4   Schwarzschild-de Sitter black hole

It seems that there is no essential difference between TAdS and TdS thermodynamically. Therefore, we are curious to study the thermal property of a black hole in de Sitter space.
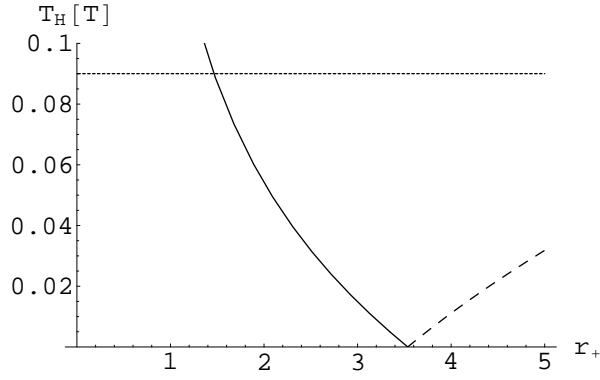
Figure 4: Temperature for SdS. Here r_+ represents r_E for the event horizon and r_C for the cosmological horizon. Here r_E (r_C) is confined to $0 < r_E \leq r_0$ ($r_0 \leq r_C < \ell$). The solid curve represents the temperature of the event horizon T_H^{ESdS}, while the dashed curve denotes the temperature of the cosmological horizon T_H^{CSdS}. The horizontal line represents the temperature $T = T_0$ of the heat bath.
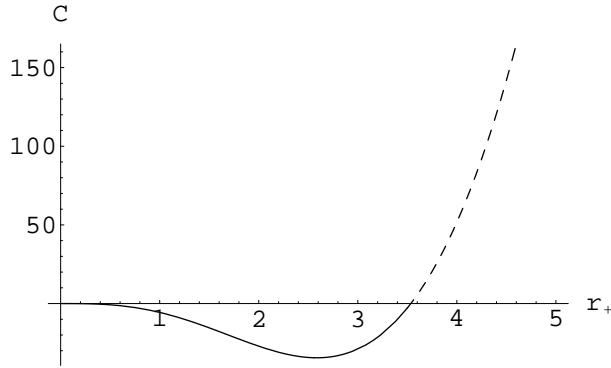

Figure 5: The graph of specific heat C^{ESdS} and C^{CSdS} for SdS. Here r_+ represents r_E for the event horizon and r_C for the cosmological horizon.

For this purpose, we introduce the Schwarzschild-de Sitter black hole in five-dimensional spacetime [16]

$$ds^2_{SdS} = -h(r)dt^2 + \frac{1}{h(r)}dr^2 + r^2 d\Omega_3^2, \tag{14}$$

where $h(r)$ is given by

$$h(r) = 1 - \frac{m}{r^2} - \frac{r^2}{\ell^2}. \tag{15}$$

In the case of $m = 0$, we have an exact de Sitter space with the largest cosmological horizon ($r_C = \ell$). However, $m \neq 0$ generates the SdS black hole. The cosmological and
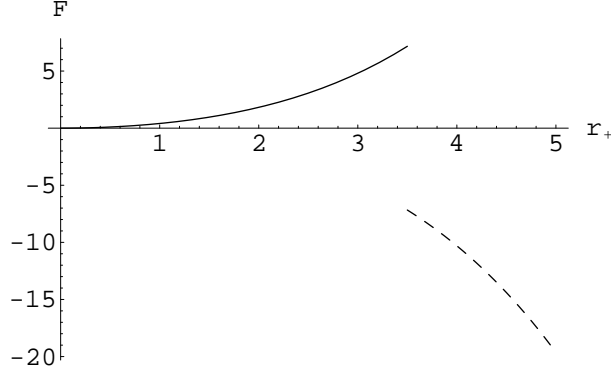
Figure 6: The plot for free energy for SdS. The solid curve represents the free energy F^{ESdS} for the event horizon, while the dashed curve denotes the free energy F^{CSdS} for the cosmological horizon. There exists a discontinuity at $r_+ = r_0$ due to the energy gab. At this point the Nariai black hole is formed.

event horizons are located at

$$r_{C/E}^2 = \frac{\ell^2}{2}\Big(1 \pm \sqrt{1 - 4m/\ell^2}\Big). \tag{16}$$

We classify three cases with $r_0 = \ell/\sqrt{2}$: $\sqrt{2m} = r_0$, $\sqrt{2m} > r_0$, and $\sqrt{2m} < r_0$. The case of $\sqrt{2m} = r_0$ corresponds to the maximum black hole and the minimum cosmological horizon (that is, Nariai black hole). Here we have $r_E = r_C = r_0$. A large black hole of $\sqrt{2m} > r_0$ is not allowed in de Sitter space. The case of $\sqrt{2m} < r_0$ corresponds to a small black hole inside the cosmological horizon. In this case a cosmological horizon is located at $r = r_C$ with $r_C \simeq \ell\sqrt{1 - m/\ell^2}$, while an event horizon is at $r = r_E$ with $r_E \simeq \sqrt{m}$. Hence, restrictions on r_E and r_C are given by

$$0 < r_E \leq r_0, \quad r_0 \leq r_C < \ell. \tag{17}$$

As the reduced mass m approaches the maximum value of $m = \ell^2/4$, the small black hole increases to the Nariai black hole at $r_E = r_0$, whereas the cosmological horizon decreases to the minimum value of $r_C = r_0$. This property contrasts to the TdS solution: there is no upper limit on the size r_C of cosmological horizon in topological de Sitter spaces.

The thermodynamic quantities for two horizons are given by [17, 18]

$$m_{E/C} = r_{E/C}^2\Big(-\frac{r_{E/C}^2}{\ell^2} + 1\Big), \quad F_{E/C} = \pm\frac{V_3 r_{E/C}^2}{16\pi G_5}\Big(\frac{r_{E/C}^2}{\ell^2} + 1\Big), \quad S_{E/C} = \frac{V_3 r_{E/C}^3}{4G_5}, \tag{18}$$

$$T_H^{E/C} = \pm\frac{1}{2\pi r_{E/C}} \mp \frac{r_{E/C}}{\pi\ell^2}, \quad E_{E/C} = \pm\frac{3V_3 m}{16\pi G_5}, \quad C_{E/C} = 3\frac{2r_{E/C}^2 - \ell^2}{2r_{E/C}^2 + \ell^2}S_{E/C},$$

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where again V_3 is the volume of a unit three-dimensional sphere Ω_3. We note here that there exists an energy gab $\Delta E = E_E - E_C$ at $r_+ = r_0$ between two horizons. This gives rise to the discontinuity in free energy.

Using the equilibrium condition $T = T_H$, we obtain a small, unstable black hole of size

$$r_u = \frac{\pi \ell^2 T}{2} \left[-1 + \sqrt{1 + \frac{8}{(2\pi \ell T)^2}} \right]$$ (19)

and a large, stable cosmological horizon of size

$$r_s = \frac{\pi \ell^2 T}{2} \left[1 + \sqrt{1 + \frac{8}{(2\pi \ell T)^2}} \right].$$ (20)

However, considering the restrictions in Eq.(17) together with $T \gg 1/\ell$, $r_s > \ell$ is not allowed in the SdS. Hence the two temperatures of T_0 and T_1 appeared in TAdS and TdS may not be relevant to the SdS phase transition.

As is shown in Fig. 4, the temperature $T_H^{E/C}$ behaves differently from the TAdS and TdS case. At $T = T_0$, we read off an unstable black hole of size $r_u = 1.46$ from a junction of line and curve but there is no stable cosmological horizon. Making use of Eqs. (17) and (18), one always finds a negative specific heat for the event horizon (ESdS) and a positive specific heat for the cosmological horizon (CSdS). See Fig. 5. The solid curve represents a negative specific heat $C^{ESdS}(\ell = 5, 0 < r_+ \leq r_0)$ for the event horizon, which shows the thermal instability clearly. At $r_+ = r_0$, we have the Nariai black hole with $T_H = 0$. The dashed curve denotes the positive specific heat $C^{CSdS}(\ell = 5, r_0 \leq r_+ < \ell)$ for the cosmological horizon, indicating the thermal stability.

Further, it is observed from Fig. 6 that one cannot obtain any critical point of $r_+ = r_1$ from the condition of $F_{E/C} = 0$. Instead we find a discontinuity $\Delta F = 3(5/4)^2 \pi$ at $r_+ = r_0$, which shows that a phase transition may occur between the event and cosmological horizon. This arises because the energy gab exists between event and cosmological horizons. As is shown in Fig. 4, we note that $T_H \to 0$, as r_+ approaches r_0. This temperature may be a candidate for the critical temperature at the ESdS-CSdS phase transition.

5   Hawking-Page phase transition

We start with the known case of SAdS. In order to understand the Hawking-Page phase transition, we introduce the generalized free energy [11]

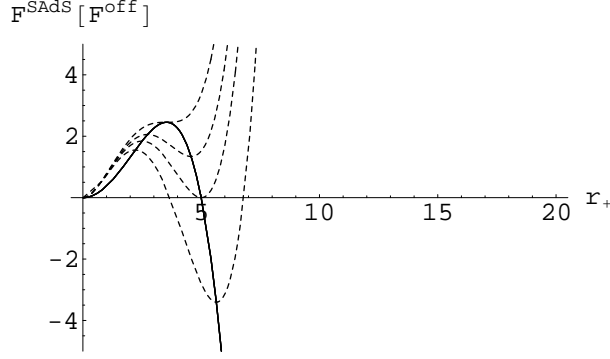$$F^{off}(\ell, r_+, T) = E(\ell, r_+) - T \cdot S(r_+)$$ (21)

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Figure 7: The process of a black hole nucleation known as the Hawking-Page phase transition. The solid curve represents the free energy $F^{SAdS}(\ell = 5, r_+)$, while the dashed curves denote the generalized free energy $F^{off}(\ell = 5, r_+, T)$ for four different temperatures: from top to bottom, $T = T_0(= 0.09)$, 0.093, $T_1(= 0.095)$, and 0.1.

which applies to any value of r_+ with a fixed temperature T. As is shown in Fig. 7, for $T = T_0$, an extremum appears at $r_+ = r_0(= r_u = r_s)$. We confirm that for $T > T_0$, there are two saddle points: a small, unstable black hole of radius r_u and a large, stable black hole of radius r_s. F is composed of a set of two saddle points for F^{off}. That is, F can be obtained from F^{off} through operation of finding saddle points: $\partial F^{off}/\partial r_+ = 0 \to T = T_H \to F = F^{off}|_{T=T_H}$. Hence, the free energy F is called the on-shell (equilibrium) free energy, whereas the generalized free energy F^{off} corresponds to the off-shell (non-equilibrium) free energy.

At this stage, we briefly review the Hawking-Page phase transition [4]. For $T_0 < T < T_1$, we find the inequality by noting the second dashed graph in Fig. 7

$$F^{off}(r_+ = 0) < F^{off}(r_+ = r_s) < F^{off}(r_+ = r_u), \tag{22}$$

which means that the saddle point at $r_+ = 0$ (thermal AdS) dominates. For $T > T_1$, the large, stable black hole dominates because $F^{off}(r_+ = r_s) < 0$. Actually, there is a change of dominance at the critical temperature $T = T_1$. In the case of $T < T_1$, the system is described by a thermal gas, whereas for $T > T_1$, it is described by a large, stable black hole. In this case, the small, unstable black hole plays a crucial role as the mediator for the transition between thermal AdS and a large, stable black hole. This is the Hawking-Page transition for a black hole nucleation in AdS space [5]. The same phase transition may occur between the zero mass de Sitter space and HTdS.

For the FAdS (FTdS) case, the small, unstable black hole is not present but the stable black hole is present for any $T > 0$. Hence the transition from $M = 0$ $(r_+ = 0)$ to
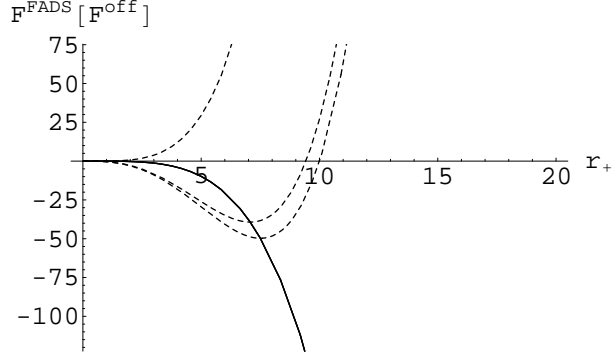
Figure 8: The off-shell transition for FAdS without a small, unstable black hole. The solid curve represents the free energy $F^{FAdS}(\ell = 5, r_+)$, while the dashed curves denote the generalized free energy $F^{off}(\ell = 5, r_+, T)$ for three different temperatures: from top to bottom $T = 0$, $T_0 (= 0.09)$, and $T_1 (= 0.095)$. The stable black holes exist as saddle points at $r_s = 7.07$ ($T = T_0$) and $r_s = 7.5$ ($T = T_1$).

$M \neq 0$ ($r_+ \neq 0$) is possible by an off-shell processes. This process is described by the off-shell free energy F^{off}. As is shown Fig. 8, at $T = 0$, the thermal AdS ($r_+ = 0$) is more favorable than $r_+ \neq 0$. The two saddle points correspond to the endpoints of the transition for $T = T_0$ and T_1. However, this does not belong to the Hawking-Page transition because there is no mediator [13]. In this case, $r_u = 0$ plays the role of a saddle point for the thermal AdS space. The same off-shell transition may occur between the zero mass de Sitter space and FTdS.

Concerning the HAdS (FTdS) case, any phase transition is unlikely to occur. This is because of the forbidden region $0 \leq r_+ < r_0$, which makes an difficulty to define the temperature T_H and the radius of unstable black hole r_u. Therefore, we could not carry out a complete analysis for the HADS and STdS.

Now we are in a position to discuss the SdS transition. First, we assume that there is a phase transition between the ESdS and CSdS at $T = T_0$, as is suggested by the SAdS (HTdS). As is shown in Fig. 9, at $T = T_0$, we obtain the location $r_u = 1.46$ of the unstable black hole from the condition of $F^{ESdS} = F^{off}$. Also we confirm this location from Fig. 4. However, the stable black hole is not present here. We note that the allowed region is confined to $0 \leq r_+ \leq \ell$. In this case, the stable black hole is located at $r_s = 8.5$ which is beyond $r_+ = \ell$. Also there exist a discontinuity at $r_+ = r_0$, which may be an obstacle to interpreting this process as a phase transition. As a result, this transition occur unlikely at $T = T_0$. Similarly, choosing $T = T_1$ leads to the similar result.

Hence, we would like to introduce the Nariai phase transition at $T = 0$. As was
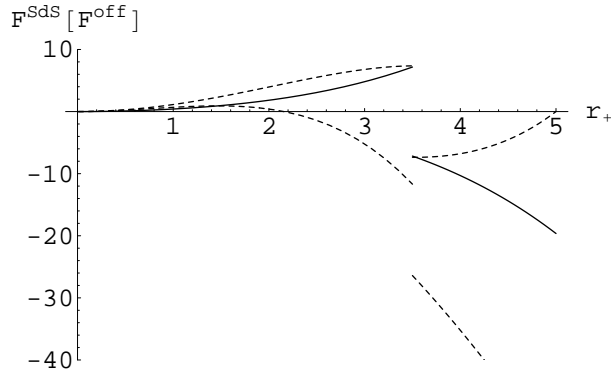
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Figure 9: The assumed transition process for SdS. The solid lines represent the free energies $F^{ESdS}(\ell = 5, 0 < r_+ \leq r_0)$ and $F^{CSdS}(\ell = 5, r_0 \leq r_+ < \ell)$. The dashed lines at top denote the generalized free energies $F^{off}(\ell = 5, 0 < r_+ \leq r_0, T = 0)$ and $F^{off}(\ell = 5, r_0 \leq r_+ \leq \ell, T = 0)$. The dashed lines at bottom show the generalized free energies at $T = T_0$. There exist discontinuities at $r_+ = r_0$ due to the energy gab between two horizons.

emphasized previously, the Nariai black hole has zero temperature. This is a case of the maximum black hole and the minimum cosmological horizon. In Fig. 9, we have $F^{ESdS} = F^{off}$ at $r_u = r_0$ and $F^{CSdS} = F^{off}$ at $r_s = r_0$. This means that the location $r_+ = r_0$ is not only the critical point of phase transition but also the position of the stable cosmological horizon. This arises because of a peculiar property of the reduced mass m in de Sitter space. As m increases to its maximum value ($m = \ell^2/4$), r_E increases to the radius of the Nariai black hole, whereas the cosmological horizon decreases to the minimum ($r_C = r_0$). We observe that at $T = 0$, $r_C = r_0$ is more favorable than $r_C \neq r_0$.

Finally, we comment on the SAdS transition that the free energies are the same at $r_+ = r_0 (= r_u = r_s)$: $F^{SAdS}(\ell = 5, r_u) = F^{SAdS}(\ell = 5, r_s)$ from Fig. 7, but the specific heat C changes from $-\infty$ to ∞ at $r_+ = r_0$ from Fig. 2. On the other hand, we observe for the Nariai transition that $F^{ESdS}(\ell = 5, r_u) \neq F^{CSdS}(\ell = 5, r_s)$, but $C^{ESdS}(\ell = 5, r_u) = C^{CSdS}(\ell = 5, r_s)$. Hence, the Nariai transition is not the Hawking-Page transition. However, the interpretation of Nariai transition may be incorrect because we have two horizons of r_E and r_C. The Killing vector $K = \partial_t$ is timelike in region $r_E < r < r_C$, while it is spacelike in the others of $0 < r < r_E$ and $r_C < r < \infty$ [7]. This may lead to the fact that the thermodynamic quantities of the event horizon are not correct to the observer outside the cosmological horizon. In this case we have to worry about the Nariai transition at $r_+ = r_0$ because of a forbidden region ($0 \leq r_E < r_0$) for the ESdS.

6 Boundary CFT and Cardy-Verlinde formula

The holographic principle means that the number of degrees of freedom associated with the bulk gravitational dynamics is determined by its boundary spacetime. The AdS/CFT correspondence represents a realization of this principle [19]. For a strongly coupled CFT with its AdS dual, one obtains the Cardy-Verlinde formula [20]. Indeed it is known that this formula holds for various kinds of asymptotically AdS spacetimes including the TAdS black holes [12]. Also it may hold for asymptotically de Sitter spacetimes including the SdS black hole and TdS spaces [16]. However, this formula is closely related to the Hawking-Page transition [11, 21]. In this section, we clarify the connection between Cardy-Verlinde formula and Hawking-Page transition. The boundary spacetimes for the (E)CFT are defined through the A(dS)/CFT correspondences [22]

$$ds^2_{CFT} = \lim_{r \to \infty} \frac{R^2}{r^2} ds^2_{TAdS} = -\frac{R^2}{\ell^2} dt^2 + R^2 d\Sigma_k^2, \tag{23}$$

$$ds^2_{ECFT} = \lim_{r \to \infty} \frac{R^2}{r^2} ds^2_{TdS} = \frac{R^2}{\ell^2} dt^2 + R^2 d\Sigma_k^2, \tag{24}$$

$$ds^2_{ECFT} = \lim_{r \to \infty} \frac{R^2}{r^2} ds^2_{SdS} = \frac{R^2}{\ell^2} dt^2 + R^2 d\Omega_3^2. \tag{25}$$

From the above, the relation between the five-dimensional bulk and four-dimensional boundary quantities is given by $E_{CFT} = (\ell/R)E$ and $T_{CFT} = (\ell/R)T_H$ where the size of boundary space R satisfies $T_{CFT} > 1/R$. As is expected, we obtain the same entropy : $S_{CFT} = S$. We note that the boundary system at high temperature is described by the CFT-radiation matter with the equation of state $p = E_{CFT}/3V_3$. Then the Casimir energy is given by $E_c = 3(E_{CFT} + pV_3 - T_{CFT}S_{CFT})$. We find the boundary thermal quantities as functions of $\hat{r} = r_{E/C}/\ell$ [23]

$$E_{CFT}^{TAdS} = \frac{3V_3\kappa\hat{r}^2(\hat{r}^2+1)}{R}, \quad T_{CFT}^{TAdS} = \frac{1}{2\pi R}\left[2\hat{r} + \frac{k}{\hat{r}}\right], \quad E_c^{TAdS} = k\frac{6V_3\kappa\hat{r}^2}{R}, \tag{26}$$

$$E_{CFT}^{TdS} = \frac{3V_3\kappa\hat{r}^2(\hat{r}^2+1)}{R}, \quad T_{CFT}^{TdS} = \frac{1}{2\pi R}\left[2\hat{r} - \frac{k}{\hat{r}}\right], \quad E_c^{TdS} = -k\frac{6V_3\kappa\hat{r}^2}{R}, \tag{27}$$

$$E_{CFT}^{E/C} = \pm\frac{3V_3\kappa\hat{r}^2(\hat{r}^2+1)}{R}, \quad T_{CFT}^{E/C} = \frac{1}{2\pi R}\left[\mp 2\hat{r} \pm \frac{1}{\hat{r}}\right], \quad E_c^{E/C} = \pm\frac{6V_3\kappa\hat{r}^2}{R} \tag{28}$$

with $\kappa = \ell^3/16\pi G_5$.

Concerning the A(dS)/CFT correspondences, we remind the reader that the boundary CFT energy (E_{CFT}) should be positive in order for it to make sense. However, one finds that $E_{CFT}^{CSdS} < 0$ for the cosmological horizon of the SdS. It suggests that the dS/CFT correspondence is not realized for this case. Also the Casimir energy (E_c) is related to the

central charge of the corresponding CFT. If it is negative, one may obtain a non-unitary CFT. In this sense HAdS, STdS, and CSdS cases may provide non-unitary conformal field theories. Furthermore, for FAdS and FTdS, there exist the forbidden region which gives rise to difficulties to define the temperature and the unstable black hole. Hence we guess that there is no realization of the (A)dS/CFT correspondences for Ricci-flat horizon.

In order to find a further implication of our study on dS/CFT correspondence, we discuss the Hawking-Page transition on the boundary system [21]. This corresponds to the transition between the confining and deconfining phase for $\mathcal{N} = 4$ Super Yang Mills gauge theory as the CFT dual to SAdS (SAdS dual). For this purpose we introduce the on-shell free energy defined by $F_{CFT}(\hat{r}) = E_{CFT} - T_{CFT}S_{CFT}$

$$F_{CFT}^{TAdS} = -\frac{V_3\kappa\hat{r}^2(\hat{r}^2 - k)}{R}, \ F_{CFT}^{TAdS} = -\frac{V_3\kappa\hat{r}^2(\hat{r}^2 + k)}{R}, \ F_{CFT}^{TAdS} = \pm\frac{V_3\kappa\hat{r}^2(\hat{r}^2 + 1)}{R} \quad (29)$$

and off-shell free energy defined by

$$F_{CFT}^{off}(\hat{r}, T) = E_{CFT} - TS_{CFT} \tag{30}$$

with the temperature of the heat reservoir T. This corresponds to the generalized free energy for the Landau-description of CFT phase transition in Ref.[11].

We guess that there is no the confining/deconfining transition for HAdS and STdS duals because of $E_c < 0$. Also this transition may not occur for CSdS dual because of $E_c < 0$ and $E_{CFT} < 0$. We show that the confining/deconfining transition is possible to occur for HTdS dual by using the Landau-description of the phase transition [11]. Finally, it seems that the dS/CFT correspondence is not realized for the SdS black hole because it contains a black hole inside the cosmological horizon. Even for the Nariai transition from the black hole to de Sitter space at $T = 0$, it is not easy to see whether the confining/deconfining transition occurs on its CFT side.

Finally, one finds the Cardy-Verlinde formulae which show the exact relation between entropy S_{CFT} and energy E_{CFT} for two cases [2] only

$$SAdS \ : \ S_{CFT} = \beta_c\sqrt{E_c(2E_{CFT} - E_c)}, \tag{31}$$

$$HTdS \ : \ S_{CFT} = \beta_c\sqrt{E_c(2E_{CFT} - E_c)} \tag{32}$$

with the inverse of critical temperature $\beta_c = 1/T_c = 2\pi R/3$ determined by $F_{CFT}^{SAdS/HTdS} = 0$.

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[2] For ESdS case, we have $S_{CFT} = (2\pi R/3)\sqrt{E_c(2E_{CFT} - E_c)}$. However, one could not convert it into Eq.(31) because there is no Hawking-Page transition at $T = T_c$.

Table 1: Summary of phase transitions for TAdS, TdS and SdS. Here $\cdot(/)$ represent the continuous (discontinuous) sign changes. HPT denotes Hawking-Page phase transition. A(d)-C means the A(dS)/CFT correspondences.

system	r_u	T_H	$E(r_+ = r_1)$	$C(r_+ = r_0)$	$F(r_+ = r_1)$	HPT	A(d)-C
HAdS	N/A	N/A	$- \cdot +$	N/A	$-$	no	no
FAdS	0	$+$	$+$	$+$	$-$	no	no
SAdS	$+$	$+$	$+$	$-\infty/\infty$	$+ \cdot -$	yes	yes
STdS	N/A	N/A	$- \cdot +$	N/A	$-$	no	no
FTdS	0	$+$	$+$	$+$	$-$	no	no
HTdS	$+$	$+$	$+$	$-\infty/\infty$	$+ \cdot -$	yes	yes
ESdS-CSdS	$+$	$+$	$+/-(r_0)$	$- \cdot +(r_0)$	$+/-(r_0)$	no	no

7 Discussion

We summarize our results in Table 1. For SAdS and HTdS, the specific heat has a pole at $r_+ = r_0$, while the free energy is maximal at $r_+ = r_0$ and zero $r_+ = r_1$. Here we have two saddle points: a small, unstable black hole and a large, stable black hole. At $T = T_1$, we find the Hawking-Page phase transition for SAdS and HTdS only. The transition point is at $r_+ = r_1$. Also the A(dS)/CFT correspondences are realized for these cases.

On the other hand, for FADS and FTdS with Ricci-flat horizon, their specific heats are monotonically increasing functions, while the free energies are monotonically decreasing functions. We confirm that there is no Hawking-Page transition (A(dS)/CFT correspondences [8]. However, for $T > 0$, there may be a transition from a zero mass background to a black hole (de Sitter space) by an off-shell process [13]. Even for taking the AdS soliton background [24], the situation is similar to the case here because there is no small, unstable black hole.

In the cases of HAdS and STdS, there exist forbidden regions. Hence, we are unable to analyze the Hawking-Page transition by making use of the specific heat and free energy. We think that the A(dS)/CFT correspondences are not realized for these.

Finally, we mention the phase transition for the SdS. At $T = 0$, the Nariai transition takes place from the black hole to de Sitter space. At this temperature, the position of $r_+ = r_0$ is not only the transition point but the location of a stable cosmological horizon. This is where the Nariai black hole is formed. However, one cannot identify its dS/CFT correspondence. On the other hand, if one considers $0 \le r_E < r_0$ to be a forbidden region, there is no Nariai transition at $T = 0$.

Acknowledgments

References

[1] R. K. Kaul and P. Majumdar, Phys. Lett. B **439** (1998) 267 [gr-qc/9801080]; R. K. Kaul and P. Majumdar, Phy. Rev. Lett, **56** (2000)5255 [gr-qc/0002040]; S. Carlip, Class. Quant. Grav. **17** (2000) 4175 [gr-qc/0005017]; T. R. Govindarajan, R.K. Kaul and V. Suneeta, Class. Quant. Grav. **18** (2001) [gr-qc/0104010]; D. Birmingham and S. Sen, Phys. Rev. D **63** (2001) 047501 [hep-th/0008051].

[2] D. J. Gross, M. J. Perry and L. G. Yaffe, Phys. Rev. D **25**, 330 (1982).

[3] J. W. York, Phys. Rev. D **33**, 2092 (1986).

[4] S. W. Hawking and D. N. Page, Commun. Math. Phys. **87**, 577 (1983).

[5] J. D. Brown, J. Creighton and R. B. Mann, Phys. Rev. D **50**, 6394 (1994) [gr-qc/9405007].

[6] E. Witten, Adv. Theor. Math. Phys. **2**, 505 (1998) [hep-th/9803131].

[7] G. W. Gibbons and S. W. Hawking, Phys. Rev. D **15**, 2738 (1977).

[8] D. Birmingham, Class. Quant. Grav. **16**, 1197 (1999) [hep-th/9808032].

[9] Y. S. Myung, Phys. Lett. B **574**, 289 (2003) [hep-th/0308191].

[10] S. Mukherji and S. S. Pal, JHEP **0205**, 026 (2002) [hep-th/0205164].

[11] L. Cappiello and W. Muck, Phys. Lett. B **522**, 139 (2001) [hep-th/0107238].

[12] R. G. Cai, Phys. Rev. D **63**, 124018 (2001) [hep-th/0102113].

[13] Y. S. Myung, Phys. Lett. B **638**, 515 (2006) [gr-qc/0603051].

[14] J. Crisostomo, R. Troncoso, and J. Zanelli, Phys. Rev. D **62**, 084013 (2000) [hep-th/0003271];

Y. S. Myung, Phys. Lett. B **624**, 297 (2005) [hep-th/0506096].

[15] V. Balasubramanian, J. de Boer, and D. Minic, Phys. Rev. D **65**, (2002) 123508 [hep-th/0110108]; R. G. Cai, Y. S. Myung, and Y. Z. Zhang, Phys. Rev. D **65**, 084019 (2002) [hep-th/0110234]; Y. S. Myung, Mod. Phys. Lett. A **16**, 2353 (2001) [hep-th/0110123]; A. M. Ghezelbach and R. B. Mann, JHEP **0201**, 005 (2002) [hep-th/0111217].

[16] R. G. Cai, Phys. Lett. B **525**, 331 (2002) [hep-th/0111093].

[17] R. G. Cai and Y. S. Myung, Phys. Rev. D **67**, 124021 (2003) [hep-th/0210272].

[18] S. Nojiri, S. D. Odintsov and S. Ogushi, Int. J. Mod. Phys. A **18**, 3395 (2003) [hep-th/0212047].

[19] J. Maldacena, Adv. Theor. Math. Phys. **2**, 231 (1998) [hep-th/9711200]; S.S Gubser, I.R. Klebanov, and A.M. Polyakov, Phys. Lett. **B428**, 105 (1998) [hep-th/9802109]; E. Witten, Adv. Theor. Math. Phys. **2**, 253 (1998) [hep-th/9802150].

[20] E. Verlinde, hep-th/0008140.

[21] Y. S. Myung, arXiv:hep-th/0604057.

[22] E. Witten, Adv. Theor. Math. Phys. **2** (1998) 505 [hep-th/9803131].

[23] Y. S. Myung, Class. Quant. Grav. **21**, 1279 (2004) [hep-th/0309179].

[24] S. Surya, K. Schleich and D. M. Witt, Phys. Rev. Lett. **86**, 5231 (2001) [hep-th/0101134].